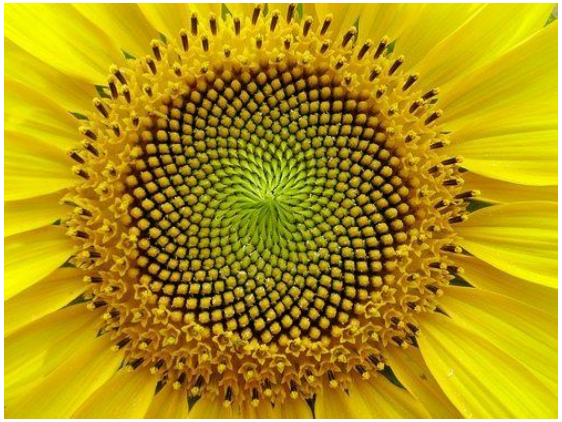
University of Ljubljana, Faculty of Computer and Information Science

Solving linear recurrences with annihilators



Prof Marko Robnik-Šikonja

Analysis of Algorithms and Heuristic Problem Solving
Edition 2024

Linear recurrence

- T(n) is a linear combination of (a small number) of nearby values T(n-1), T(n-2),…
- Example: Fibonacci
  $T(n) = T(n-1) + T(n-2)$
  $T(0) = 0$
  $T(1) = 1$
- Solution: in general, a combination of polynomials and exponential functions
- How to solve these equations?

- Jeff Erickson: Models of Computation (Solving Recurrences, Appendix II), 2019
  available at http://jeffe.cs.illinois.edu/teaching/algorithms/

Operators

- Operators are higher order functions, taking other functions as their arguments
- For example, integral $\int f(x)dx$ or differential $\frac{df}{dx}$ are operators
- In solving recurrences we need three operators
  - sum (f+g)(n) = f(n) + g(n)
  - scale (a·f)(n) = a·(f(n))
  - shift (Ef)(n) = f(n+1)
- Scale and shift are linear (can be distributed over sums)
- We combine operators and get compound operators

Manipulation of operators

- Compound manipulators behave as polynomials over variable E

Operator manipulation

Operator	Definition
addition	$(f+g)(n) := f(n) + g(n)$
subtraction	$(f-g)(n) := f(n) - g(n)$
multiplication	$(\alpha \cdot f)(n) := \alpha \cdot (f(n))$
shift	$\boldsymbol{E} f(n) := f(n+1)$
k-fold shift	$\boldsymbol{E}^k f(n) := f(n+k)$
composition	$(\boldsymbol{X}+\boldsymbol{Y}) f := \boldsymbol{X} f + \boldsymbol{Y} f$
	$(\boldsymbol{X}-\boldsymbol{Y}) f := \boldsymbol{X} f - \boldsymbol{Y} f$
	$\boldsymbol{X}\boldsymbol{Y} f := \boldsymbol{X}(\boldsymbol{Y} f) = \boldsymbol{Y}(\boldsymbol{X} f)$
distribution	$\boldsymbol{X}(f+g) = \boldsymbol{X} f + \boldsymbol{X} g$

Annihilators

- Annihilator is a nontrivial operator transforming function to zero.
- Multiplication by zero is a trivial operator, which we don't take into account.
- Every compound operator annihilates a specific class of functions
- Every function composed of polynomial and exponential functions has a unique (minimal) annihilator
- The goal: find annihilators from different class of functions.

Annihilator behaviour

Operator	Function annihilated
E-1	α
E-a	αa^n
(E-a)(E-b)	$\alpha a^n + \beta b^n$; if $a \neq b$
$(E-a_0)\,(E-a_1)\ldots(E-a_k)$	$\sum_{i=0}^{k} \alpha_i a_i^n$; if $a_i \neq a_j$ for all i, j
$(E-1)^2$	$\alpha n + \beta$
$(E-a)^2$	$(\alpha n + \beta)a^n$
$(E-a)^2(E-b)$	$(\alpha n + \beta)a^n + \gamma b^n$; if $a \neq b$
$(E-a)^d$	$(\sum_{i=0}^{d-1} \alpha_i n^i)a^n$

- If ***X*** annihilates *f*, then ***X*** also annihilates ***E*** *f*.
- If ***X*** annihilates both *f* and *g*, then ***X*** also annihilates *f* ± *g*.
- If ***X*** annihilates *f*, then ***X*** also annihilates αf, for any constant α.
- If ***X*** annihilates *f* and ***Y*** annihilates *g*, then ***XY*** annihilates *f* ± *g*.

Annihilating recurrences

- To solve linear recurrences (remember their solutions are composed of polynomials and exponentials) one has to annihilate them.

  1. Write the recurrence in operator form
  2. Extract an annihilator for the recurrence
  3. Factor the annihilator (if necessary) (and possible)
  4. Extract the generic solution from the annihilator
  5. Solve for coefficients using base cases (if known)

Generating functions

- Generating functions are a generalization of annihilators.
- General tool for combinatorics and counting.

- Recommended further reading:

Robert Sedgewick and Philippe Flajolet. *An introduction to the analysis of algorithms*. Pearson, 1996.

(also 2nd edition, 2013)