Vaje 5. teden

5 c) $\sqrt[5]{-32i}$

$z = -32i$

$|z| = 32$

$\varphi = \pi$

$z = 32e^{i\pi}$

$z_k = \sqrt[5]{z} = \sqrt[5]{32}\, e^{i\frac{\pi + 2k\pi}{5}}$; $k \in 0, \dots, 4$

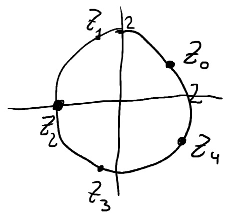
$z_0 = 2e^{i\frac{\pi}{5}}$

$z_1 = 2e^{i\frac{3\pi}{5}}$

$z_2 = 2e^{i\frac{5\pi}{5}} = 2e^{i\pi} = -2$

$z_3 = 2e^{i\frac{7\pi}{5}}$

$z_4 = 2e^{i\frac{9\pi}{5}}$



d) $\sqrt[3]{-1+i\sqrt{3}}$

$z = -1 + i\sqrt{3}$

$|z| = \sqrt{(-1)^2 + \sqrt{3}^2} = \sqrt{4} = 2$

$\arctan \varphi = \frac{\sqrt{3}}{-1} = -\sqrt{3}$

$\varphi_1 = -\frac{\pi}{3}$ $\quad \boxed{\varphi_2 = \frac{2\pi}{3}}$

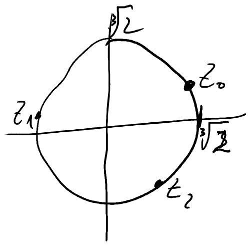
$z = 2e^{i\frac{2\pi}{3}}$

$z_k = \sqrt[3]{2e^{i\frac{2\pi}{3}}} = \sqrt[3]{2}\, e^{i\frac{\frac{2\pi}{3} + 2k\pi}{3}}$ $\quad k = 0, 1, 2$

$z_0 = \sqrt[3]{2}\, e^{i\frac{2\pi}{9}}$

$z_1 = \sqrt[3]{2}\, e^{i\frac{8\pi}{9}}$

$z_2 = \sqrt[3]{2}\, e^{i\frac{14\pi}{9}}$



1. Nariši naslednjo podmnožico v $\mathbb{C}$:

$$A = \{z \in \mathbb{C} \,;\, 1 < |z| < 4,\ 0 \le \arg(z) < \pi/4,\ \mathrm{Im}(z) < 2\}$$

$z = x + yi = re^{i\varphi}$

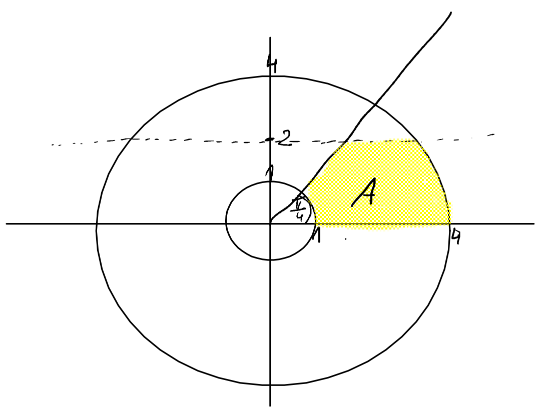
$1 < |z| < 4$

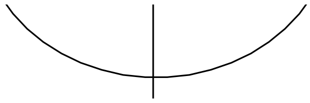
$1 < \sqrt{x^2+y^2} < 4 \quad /^2$

$1 < x^2 + y^2 < 4^2 \Rightarrow r \in (1, 4)$

$\varphi \in [0, \frac{\pi}{4})$

$y < 2$





Z območjem A naredimo naslednjo transformacijo: $z = |z|e^{i\varphi}$

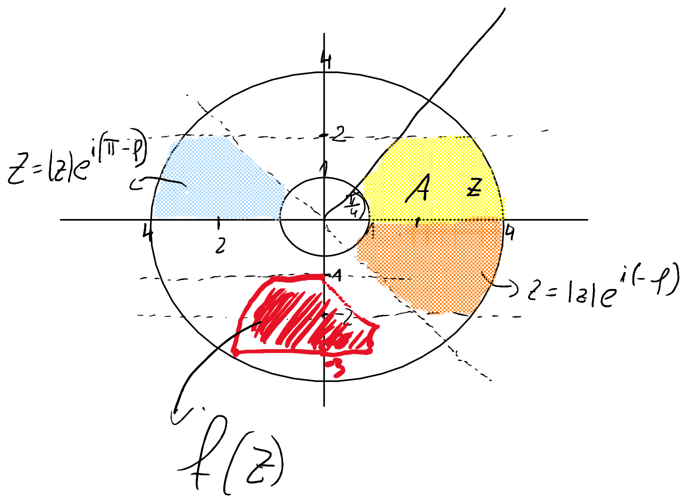
(a) prezrcalimo ga preko realne osi, $z \to |z|e^{i(-\varphi)}$

(b) zavrtimo ga okoli števila 0 za kot π, $z \to z \cdot e^{i\pi}$

(c) premaknemo ga za 2 v desno in 3 navzdol. $z \to z + 2 + 3i$

Zapiši predpis $z \mapsto f(z)$, ki opravi to kompleksno transformacijo. Nariši tudi $f(A)$ in ugotovi, kam se preslika število $1+i$.

$$f(z) = |z|e^{i((-\varphi)+\pi)} + 2 + 3i = |z|e^{i(-\varphi+\pi)} + 2 + 3i$$



$z = 1+i$

$|z| = \sqrt{2}$

$\varphi = \frac{\pi}{4}$

$z = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$$\underset{1+i}{\overset{\shortparallel}{z}} \xrightarrow{a)} \underset{1-i}{\overset{\shortparallel}{\sqrt{2}\, e^{i\left(\frac{\pi}{4}\right)}}} \xrightarrow{b)} \underset{-1+i}{\overset{\shortparallel}{\sqrt{2}\, e^{i\left(-\frac{\pi}{4}+\pi\right)}}} \xrightarrow{c)} \underset{1-2i}{\overset{\shortparallel}{\sqrt{2}\, e^{i\left(\frac{3\pi}{4}\right)} + 2 - 3i}}$$

$f(1+i) = 1 - 2i$

2. Zaporedje je dano s predpisom

$$a_n = \frac{2n-1}{n+3}.$$

(a) Izračunaj nekaj členov in nariši graf zaporedja. Pomagaj si z grafom funkcije $y = \frac{2x-1}{x+3}$.

(b) Ali je zaporedje naraščajoče, padajoče? Prepričaj se z računom.

(c) Prepričaj se, da je zaporedje konvergentno in izračunaj njegovo limito a. Od katerega n dalje ležijo vsi členi tega zaporedja znotraj intervala $\left(a - \frac{1}{4}, a + \frac{1}{4}\right)$?

$a_1 = \frac{1}{4}$

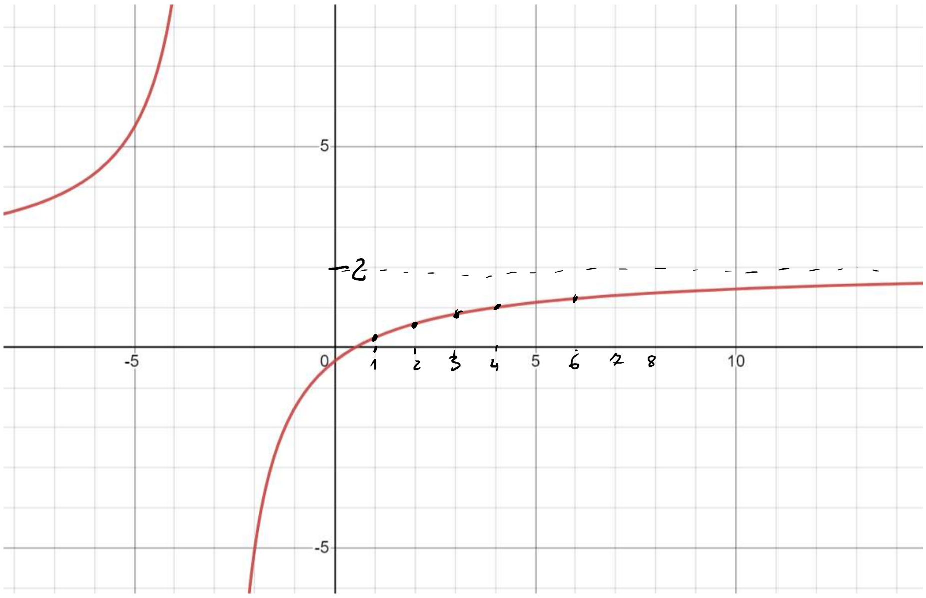
$a_2 = \frac{3}{5}$

$a_3 = \frac{5}{6}$

$a_4 = \frac{7}{7}$

$a_5 = \frac{9}{8}$

$a_6 = \frac{11}{9}$



b) a_n je NARAŠČAJOČE

$$a_{n+1} > a_n$$

$$\frac{2(n+1)-1}{(n+1)+3} > \frac{2n-1}{n+3}$$

$$\frac{2n+1}{n+4} > \frac{2n-1}{n+3} \quad / \cdot \underset{>0}{(n+4)} \cdot \underset{>0}{(n+3)}$$

$$(2n+1)(n+3) > (2n-1)(n+4)$$

$$\cancel{2n^2} + \cancel{6n} + \cancel{n} + 3 > \cancel{2n^2} + \cancel{8n} - \cancel{n} - 4$$

$3 > -4$ ✓ JE NARAŠČAJOČE

c) NARAŠČAJOČE + OMEJENO ⇒ KONVERGENTNO

$$a_n < 2$$

$$\frac{2n-1}{n+3} < 2 \quad / \cdot \underset{>0}{(n+3)} \quad \Rightarrow \text{JE OMEJENO}$$

$$2n - 1 < 2n + 6$$

$-1 < 6$ ✓

IŠČEMO NAJMANJŠI n, ZA KATEREGA VELJA

$$a - \frac{1}{4} < a_n < a + \frac{1}{4} \qquad a = 2$$

$$\underbrace{\frac{7}{4} < a_n}\, < \frac{9}{4}$$

→ VEDNO RES, KER JE < 2

$$\frac{7}{4} < \frac{2n-1}{n+3} \quad / \cdot 4 \cdot (n+3)$$

$7(n+3) < 4(2n-1)$

$7n+21 < 8n-4$

$16 < n \longrightarrow$ OD 17 ČLENA NAPREJ

3. Zaporedje (a_n) je dano rekurzivno

$$a_0 = 3,\ a_{n+1} = \sqrt{2+a_n}.$$

(a) Preveri, da je zaporedje (a_n) padajoče in velja $a_n \geq 2$ za vsako naravno število n.

(b) Koliko je $\lim_{n\to\infty} a_n$?

$a_0 = 3$

$a_1 = \sqrt{5}$

$a_2 = \sqrt{2+\sqrt{5}}$

$a_3 = \sqrt{2+\sqrt{2+\sqrt{5}}}$

$\vdots$

$a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$

$\sqrt{2+a_n} \leq a_n$ $\quad$ $\nearrow^2$ VSI ČLENI SO POZITIVNI

$2+a_n \leq a_n^2$

$0 \leq a_n^2 - a_n - 2$

$x^2 - x - 2 \geq 0$

$(x-2)(x+1) \geq 0$ $\quad$ $x \in (-\infty, -1] \cup [2, \infty)$

NAJPREJ REŠIMO TO

KER SO $a_n \geq 2$ VELJA $a_{n+1} \leq a_n$ ZA VSAK n

$a_n \geq 2 \quad \forall n$

MATEMATIČNA INDUKCIJA

BAZA $n=1$: $a_1 = \sqrt{5} \geq 2$ ✓

INDUKCIJSKA PREDPOSTAVKA : $a_n \geq 2$

DOKAŽEMO $a_{n+1} \geq 2$

$$a_{n+1} = \sqrt{2+\underset{\geq 2}{a_n}} \geq \sqrt{4} = 2$$

$a_{n+1} \geq 2$ ✓ ZA VSAK n

b) LIMITA $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \underbrace{L}_{\text{OZNAKA}}$

$a_{n+1} = \sqrt{2+a_n}$ ZA VSAK n

$\lim_{n\to\infty}$ $\quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n}$

$\lim_{n\to\infty}$ $\hookrightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n}$

$$\lim_{n\to\infty} a_{n+1} = \sqrt{2+\lim_{n\to\infty} a_n}$$

$$L = \sqrt{2+L} \quad \nearrow^2$$

$$L^2 = 2 + L$$

$$L^2 - L - 2 = 0$$

$$(L-2)(L+1) = 0$$

$$\left.\begin{array}{l} L_1 = -1 \\ L_2 = 2 \end{array}\right\}$$ KANDIDATA $\Rightarrow$ KER SO VSI $a_n \geq 2$ VELJA: $\boxed{L=2}$