Mathematical modelling, Exam 3

22. 8. 2019

1. (a) Construct any non-diagonal 3×2 matrix A whose singular values are 2 and 1.

   (b) Find the Moore-Penrose inverse A^+.

   (c) Let $b \in \mathbb{R}^2$. Describe the property uniquely characterizing point $A^+ \cdot b$ with respect to the system $Ax = b$.

   Solution.

   (a) One possible solution is

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \\ 0 & 0 \end{bmatrix}.$$

   (b) The Moore-Penrose inverse of A is

$$A^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}.$$

   (c) If the system $Ax = b$ is solvable, then the vector $A^\dagger b$ is the solution of the system of the smallest norm among all solutions. Otherwise the vector $A^\dagger b$ is the unique solution of the system of the smallest norm w.r.t. the least squares method, i.e.,

$$\left\| A(A^\dagger b) - b \right\| = \min \left\{ \|Ax - b\| : x \in \mathbb{R}^2 \right\}.$$

2. Two surfaces in the upper halfspace $z > 0$ are given by the following equations:

$$\Pi : x^2 + y^2 = \frac{z^2}{2} \qquad \Sigma : x^2 + y^2 = z.$$

   Curve γ is the intersection of surfaces Π and Σ. Let $P = (1, 1, 2) \in \gamma$.

   (a) Find the angle at which the surfaces intersect at P.

(b) Find the line tangent to γ at P.

(c) Find the plane that is tangent to Σ at $(1, 2, 5)$.

Solution.

(a) The angle at which the surfaces intersect at P is the angle between the normals to their tangent planes at the point P, i.e., between their gradients at the point P:

$$\begin{aligned}
&\arccos\left(\frac{\langle(\operatorname{grad}\Pi)(P), (\operatorname{grad}\Sigma)(P)\rangle}{\|\operatorname{grad}\Pi)(P)\|\,\|\operatorname{grad}\Sigma)(P)\|}\right)\\
&= \arccos\left(\frac{\langle(2x, 2y, -z)(P), (2x, 2y, -1)(P)\rangle}{\|(2x, 2y, -z)(P)\|\,\|(2x, 2y, -1)(P)\|}\right)\\
&= \arccos\left(\frac{\langle(2, 2, -2), (2, 2, -1)\rangle}{\|(2, 2, -2)\|\,\|(2, 2, -1)\|}\right) = \arccos\left(\frac{10}{\sqrt{12}\sqrt{9}}\right)\\
&= \arccos\left(\frac{5}{3\sqrt{3}}\right) \approx 0.28.
\end{aligned}$$

(b) The intersection of the surfaces satisfies

$$x^2 + y^2 = \frac{(x^2 + y^2)^2}{2} \quad \Rightarrow \quad 2 = x^2 + y^2 \quad \Rightarrow \quad z = 2.$$

So this is a circle with the parametrization

$$\gamma(t) = \left(\sqrt{2} \cdot \cos t, \sqrt{2} \cdot \sin t, 2\right).$$

The tangent to this circle in the point $(1, 1, 2)$, which corresponds to $t = \frac{\pi}{4}$, is

$$\begin{aligned}
\ell(\lambda) &= (1, 1, 2) + \lambda \cdot \left(\gamma'\left(\frac{\pi}{4}\right)\right)\\
&= (1, 1, 2) + \lambda \cdot \left(\left(-\sqrt{2} \cdot \sin t, \sqrt{2} \cdot \cos t, 0\right)\left(\frac{\pi}{4}\right)\right)\\
&= (1, 1, 2) + \lambda \cdot (-1, 1, 0).
\end{aligned}$$

(c) The tangent plane to Σ at $Q := (a, b, c) = (1, 2, 5)$, which is given implicitly by the equation

$$F(x, y, z) = 0,$$

where

$$F(x, y, z) := x^2 + y^2 - z,$$

is determined by

$$\begin{aligned}
0 &= \left(\frac{\partial F}{\partial x}(Q)\right) \cdot (x-a) + \left(\frac{\partial F}{\partial y}(Q)\right) \cdot (y-b) + \left(\frac{\partial F}{\partial z}(Q)\right) \cdot (z-c) \\
&= ((2x)(Q)) \cdot (x-1) + ((2y)(Q)) \cdot (y-2) + ((-1)(Q)) \cdot (z-5) \\
&= 2(x-1) + 4(y-2) - (z-5).
\end{aligned}$$

3. Solve the following exact differential equation $2xy + (x^2 + 3y^2)y' = 0$.

   Solution. The DE is of the form

   $$2xy\ dx + (x^2 + 3y^2)\ dy = 0.$$

   It is indeed exact, since

   $$\frac{d(2xy)}{dy} = \frac{d(x^2 + 3y^2)}{dx} = 2x.$$

   We have that

   $$\int 2xy\ dx = x^2 y + C(y),$$
   $$\int (x^2 + 3y^2)\ dx = x^2 y + y^3 + D(x),$$

   where $C(y)$ and $D(x)$ are functions of y and x. Hence, the solution of the DE is a family of functions

   $$u(x, y, K) = x^2 y + y^3 + K,$$

   where $K \in \mathbb{R}$ is a constant.

4. Solve the differential equation $y'' + 9y = 2x^2 - 1$. with the initial condition $y(0) = y'(0) = 1$.

   Solution. First we solve the homogeneous part of the DE:

   $$y'' + 9y = 0.$$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 + 9 = (\lambda - 3i)(\lambda + 3i)$$

with zeroes $\lambda_1 = 3i$, $\lambda_2 = -3i$. Hence, the solution of the homogeneous part is

$$y_h(x) = Ce^{3ix} + De^{-3ix},$$

where $C, D \in \mathbb{C}$ are constants. Another way of expressing all solutions of the DE is

$$y_h(x) = C\cos(3x) + D\sin(3x),$$

where $C, D \in \mathbb{C}$ are constants.

To obtain a particular solution we can try with the form

$$y_p(x) = ax^2 + bx + c \quad \Rightarrow \quad y_p'(x) = 2ax + b \quad \Rightarrow \quad y_p''(x) = 2a. \tag{1}$$

Plugging (1) into the DE we obtain

$$2a + 9(ax^2 + bx + c) = 2x^2 - 1. \tag{2}$$

Comparing the coefficients at $x^2, x, 1$ on both sides of (2) we obtain a system

$$9a = 2, \quad 9b = 0, \quad 2a + 9c = -1,$$

with the solution

$$a = \frac{2}{9}, \quad b = 0, \quad c = -\frac{13}{81}.$$

Hence, the general solution of the DE is

$$y(x) = y_h(x) + y_p(x) = C\cos(3x) + D\sin(3x) + \frac{2}{9}x^2 - \frac{13}{81}.$$

The one satisfying the initial conditions

$$y(0) = C - \frac{13}{81} = 1,$$
$$y'(0) = 3D = 1,$$

is the one with

$$C = \frac{94}{81}, \quad D = \frac{1}{3}.$$

So the final solution is

$$y(x) = \frac{94}{81}\cos(3x) + \frac{1}{3}\sin(3x) + \frac{2}{9}x^2 - \frac{13}{81}.$$