Mathematical modelling

2. 7. 2018

1. Given points $(0,0,0), (2,0,1)$, $(1,1,0)$ and $(0,1,1)$ in $\mathbb{R}^3$ your goal is to find the plane $ax+by+cz=1$ which best fits the given points.

   (a) Write down the system of equations for the parameters a, b and c given by the point coordinates.

   (b) Compute the Moore-Penrose inverse A^+ of the coefficient matrix A of the system.

   (c) Finally, write down the equation of the best-fit plane.

   Solution:

   (a) The coefficient matrix of the system is

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

   (b) Since $A^TA = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ is invertible with determinant 9 and

$$(A^TA)^{-1} = \frac{1}{9}\begin{bmatrix} 3 & 0 & -3 \\ 0 & 6 & -3 \\ -3 & -3 & 9 \end{bmatrix},$$

$$A^+ = (A^TA)^{-1}A^T = \begin{bmatrix} 0 & 1/3 & 1/3 & -1/3 \\ 0 & -1/3 & 2/3 & 1/3 \\ 0 & 1/3 & -2/3 & 2/3 \end{bmatrix}$$

   (c) Finally $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^+ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$ and the equation of the best fit plane is $1/3x + 2/3y + 1/3z = 1$.

2. A parametric surface is given by

$$f(u,v) = \begin{bmatrix} u\cos(v) \\ u\sin(v) \\ \log(u^2+1) \end{bmatrix}, \quad u \geq 0, 0 \leq v \leq 2\pi$$

(a) Find the equation of the tangent plane at the point $u = 1, v = 1$.

(b) Find the coordinate curves through the point $u = 1, v = 1$.

(c) Sketch several other coordinate curves. How can you best describe the shape of the surface?

Solution:

(a) The tangent plane is given by the linear approximation of f at $(1,1)$.

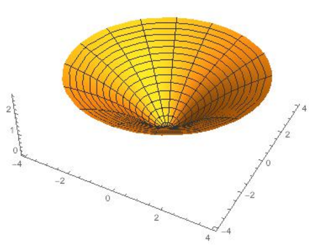
$$Df = \begin{bmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \\ \frac{2u}{u^2+1} & 0 \end{bmatrix},$$

$$L_{(1,1)}(u,v) = f(1,1) + Df(1,1)\begin{bmatrix} u-1 \\ v-1 \end{bmatrix} = \begin{bmatrix} \cos 1 \\ \sin 1 \\ log 2 \end{bmatrix} + \begin{bmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} u-1 \\ v-1 \end{bmatrix}.$$

(b) The coordinate curves through $f(1,1)$ are given by

$$f(1,v) = \begin{bmatrix} \cos v \\ \sin v \\ \log 2 \end{bmatrix} \quad \text{and} \quad f(u,1) = \begin{bmatrix} u\cos 1 \\ u\sin 1 \\ \log(u^2+1) \end{bmatrix}$$

(c) (2 bonus points) The coordinate curve $f(u_0, v)$ is a circle in the plane $z = \log(u_0^2 + 1)$ for all $u_0 > 0$, the surface is a surface of revolution obtained by rotating the curve $z = \log(x^2+1)$ in the

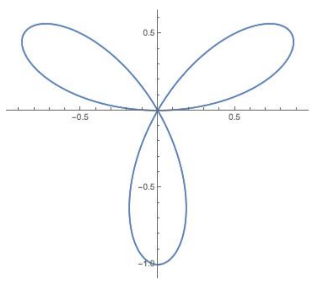


(x,z) plane around the z-axis.

3. For the curve in polar coordinartes $r(\varphi) = \sin(3\varphi)$

   (a) sketch the curve,
   (b) find the area enclosed by one loop,
   (c) express the curve in parametric form,
   (d) find the equation of the tangent to the curve at the point given by $\varphi = \pi/6$.

   Solution:

   (a) 

   (b) The area enclosed by one loop:

$$P = \frac{1}{2}\int_0^{\pi/3} \sin^2(3\varphi)\, d\varphi = \frac{1}{4}\int_0^{\pi/3} 1 - \cos(6\varphi)\, d\varphi = \frac{\pi}{12}.$$

   (c) Since $x = r\cos\varphi$ and $y = r\sin\varphi$, the standard parametrization is $f(\varphi) = \begin{bmatrix} \sin(3\varphi)\cos(\varphi) \\ \sin(3\varphi)\sin(\varphi) \end{bmatrix}$.

   (d) Since $\dot{f}(\varphi) = \begin{bmatrix} 3\cos(3\varphi)\cos(\varphi) + \sin(3\varphi)\sin(3\varphi) \\ 3\cos(3\varphi)\sin(\varphi) - \sin(3\varphi)\cos(3\varphi) \end{bmatrix}$, the tangent at $\varphi = \pi/6$ is

$$L_{\pi/6}(t) = f(\pi/6) + \dot{f}(\pi/6)t = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix} t$$

   (2 bonus points for the precise equation)

4. Given the differential eqaution $\ddot{x} + 5\dot{x} + 4x = t + 1$

   (a) find the general solution of the corresponding homogeneous equation,

(b) find the general solution of the given nonhomogeneous equation,

(c) find the solution satisying the initial conditions $x(0) = 3$ and $\dot{x}(0) = 0$.

Solution:

(a) The homogenous equation $\ddot{x} + 5\dot{x} + 4x = 0$ has characteristic polynomial

$$\lambda^2 + 5\lambda + 4 = (\lambda + 4)(\lambda + 1),$$

so the general solution is $x_h(t) = C_1 e^{-4t} + C_2 e^{-t}$.

(b) The solution is of the form $x(t) = x_h(t) + x_p(t)$, where $x_p(t) = At + b$. Inserting this into the equation we obtain $x_p(t) = 1/4t - 1/16$.

(c) From the system $x(0) = C_1 + C_2 - 1/16 = 3, \dot{x}(0) = -4C_1 - C_2 + 1/4 = 0$, we obtain C_1 and C_2 (2 bonus points for the precise values).