Mathematical Modelling Exam

30. 06. 2021

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. We are given the matrix A and the vector b:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

(a) What is the rank of A?

(b) Compute one generalized inverse of A.

(c) Determine all solutions of the system $Ax = b$.

Solution.

(a) The left upper 2×2 submatrix has determinant -1 and hence the first two columns are linearly independent. The third column is the sum of the first two and so $\operatorname{rank} A = 2$.

(b) We choose an invertible 2×2 submatrix of A, replace it with its transposed inverse, replace the other entries with 0s and transpose the matrix obtained:

$$G = \begin{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\right)^T & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) The candidate for the solution is

$$Gb = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

We can check that $A(Gb) = b$ and hence Gb is really a solution of the system. All solutions are of the form

$$\begin{aligned} Gb + (GA - I)z &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 + z_3 \\ -1 + z_3 \\ -z_3 \end{bmatrix}, \end{aligned}$$

where $z_1, z_2, z_3 \in \mathbb{R}$.

2. Using one step of Newton's method approximate the solution of the system

$$\sin x + \cos y + e^{xy} = \arctan(x+y) - xy = 0,$$

with the initial approximation $(x_0, y_0) = (0, 0)$.

Solution. We define a vector function $f : \mathbb{R}^2 \to \mathbb{R}^2$ by the rule

$$f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} \sin x + \cos y + e^{xy} \\ \arctan(x+y) - xy \end{bmatrix}.$$

We are approximating the zero $f(x,y) = 0$ of f using one step of Newton's method with the with the initial approximation $(x_0, y_0) = (0, 0)$. The Jacobian $Jf(x,y)$ of f is

$$Jf(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos x + y e^{xy} & -\sin y + x e^{xy} \\ \frac{1}{(x+y)^2+1} - y & \frac{1}{(x+y)^2+1} - x \end{bmatrix}.$$

So

$$\begin{aligned} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \left(Jf(x_0,y_0)\right)^{-1} f(x_0,y_0) \\ &= -\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= -\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}. \end{aligned}$$

3. Let

$$r(t) = (2\sin(2t), 2\cos(2t), 3t)$$

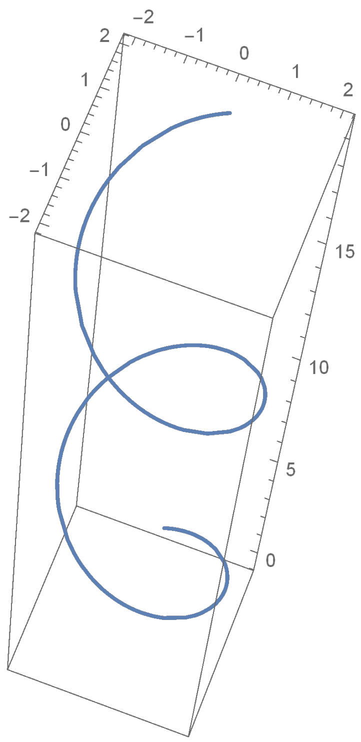
be the curve in parametric coordinate with $t \in [0, 2\pi]$.

(a) Sketch the curve in $\mathbb{R}^3$.

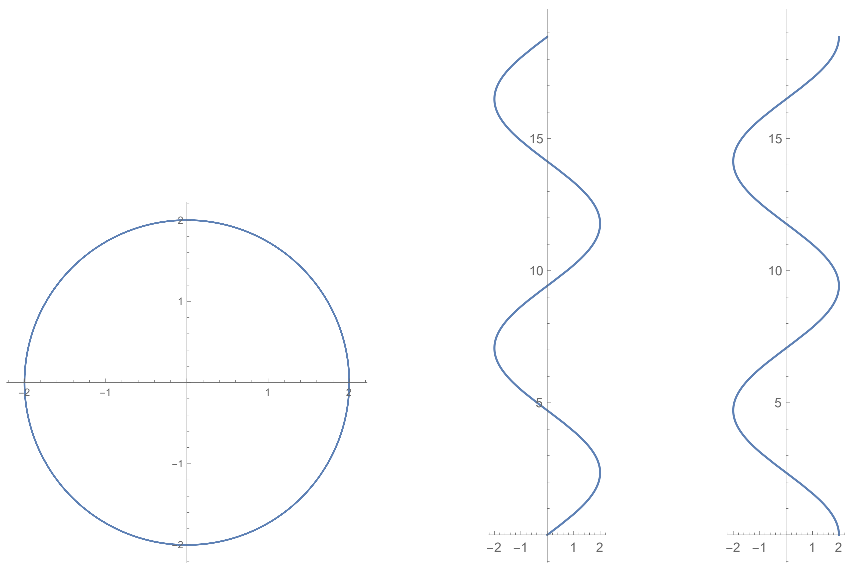
(b) Sketch all three projections of the curve in the xy-, xz- and yz-coordinate planes.

(c) Compute the arc length of the curve.

Solution. The sketch of the curve is the following:



The sketch of the projections to xy-, xz- and yz-planes are:



The arc length is the following:

$$\begin{aligned}
\int_0^{2\pi} \|r'(t)\|\,dt &= \int_0^{2\pi} \|(4\cos(2t), -4\sin(2t), 3)\|\,dt \\
&= \int_0^{2\pi} \sqrt{16\cos^2(2t) + 16\sin^2(2t) + 9}\,dt \\
&= \int_0^{2\pi} \sqrt{25}\,dt = 10\pi.
\end{aligned}$$

4. Solve the differential equation

$$y'' - 4y' + 5y = 8\cos x.$$

Find a solution to this DE which has a local extremum in the point $(0, 2)$.

Solution. First we solve the homogeneous part of the DE. The characteristic polynomial is $\lambda^2 - 4\lambda + 5$ with zeroes

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i.$$

So, two linearly independent solutions to the homogeneous part are

$$y_1(t) = e^{(2+i)t} = e^{2t}e^{it} \qquad \text{and} \qquad y_2(t) = e^{(2-i)t} = e^{2t}e^{-it}.$$

Two *real* linearly independent solutions are

$$y_{\mathbb{R},1}(t) = e^{2t}\cos t \qquad \text{and} \qquad y_{\mathbb{R},2}(t) = e^{2t}\sin t.$$

A general solution of the homogeneous part is

$$y_h(t) = Ay_{\mathbb{R},1}(t) + By_{\mathbb{R},2}(t), \qquad \text{where} \quad A, B \in \mathbb{R}.$$

To find one particular solution of the DE we can use the form

$$y_p(t) = C\cos t + D\sin t.$$

Plugging this form into the initial DE we obtain

$$(-C\cos t - D\sin t) - 4(-C\sin t + D\cos t) + 5(C\cos t + D\sin t) = 8\cos x.$$

Comparing the coefficients at $\cos t$ and $\sin t$ on both sides we obtain the system

$$4C - 4D = 8 \qquad \text{and} \qquad 4D + 4C = 0,$$

with the solution $C = -D = 1$. So a general solution to the DE is

$$y(t) = y_h(t) + y_p(t) = e^{2t}(A\cos t + B\sin t) + \cos t - \sin t.$$

The solution with a local extremum in the point $(0, 2)$ is determined by $y(0) = 2$ and $y'(0) = 0$:

$$\begin{aligned} 2 &= y(0) = A + 1, \\ 0 &= y'(0) = \big(2e^{2t}(A\cos t + B\sin t) + e^{2t}(-A\sin t + B\cos t) - (\sin t + \cos t)\big)(0) \\ &= 2A + B - 1. \end{aligned}$$

So $A = 1$ and $B = -1$.