Ime in priimek

Vpisna številka

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2	
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Σ	

Linearna algebra: 1. kolokvij

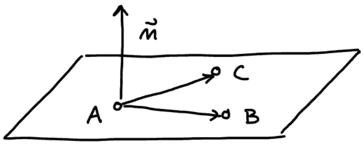
17. april 2024

Čas pisanja: 75 minut. Dovoljena je uporaba dveh listov velikosti A4 z obrazci. Uporaba elektronskih pripomočkov ni dovoljena. Rezultati bodo objavljeni na *ucilnica.fri.uni-lj.si*. **Vse odgovore dobro utemelji!**

1. naloga (35 točk)

Dane so premica $p : \vec{x} = t \cdot [1,2,2]^T + [-1,1,2]^T$ ter točke $A(1,-1,1)$, $B(-2,2,1)$ in $C(-4,-1,-4)$.

a) (10) Poišči enačbo ravnine R, na kateri ležijo točke A, B in C.



$$\vec{n} \parallel \vec{AB} \times \vec{AC} = \begin{bmatrix}-3\\3\\0\end{bmatrix} \times \begin{bmatrix}-5\\0\\-5\end{bmatrix} = \begin{bmatrix}-15\\-15\\15\end{bmatrix}$$

$$\vec{n} = \begin{bmatrix}1\\1\\-1\end{bmatrix} \qquad \vec{n} \cdot \vec{r}_A = \begin{bmatrix}1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\1\end{bmatrix} = 1-1-1=-1$$

$$R: \ x+y-z=-1$$

b) (5) Poišči presečišče premice p in ravnine R.

$$p: \begin{bmatrix}t\\2t\\2t\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix} = \begin{bmatrix}t-1\\2t+1\\2t+2\end{bmatrix} \in R \Rightarrow t \cancel{-1} + \cancel{2t} \cancel{+1} - \cancel{2t} - 2 = -1$$

$$t-2=-1$$

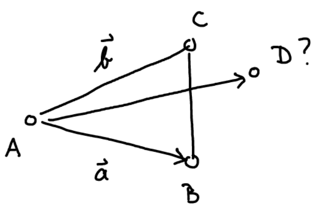
$$t=1 \qquad T(0,3,4)$$

c) (10) Določi enačbo premice q skozi točki A in B.

$$\vec{q} \parallel \vec{AB} = \begin{bmatrix}-3\\3\\0\end{bmatrix} \rightarrow \vec{q} = \begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad q: \begin{bmatrix}1\\-1\\1\end{bmatrix} + t\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$A(1,-1,1)$

d) (10) Ali točka $D(-2,0,-1)$ leži znotraj trikotnika $\triangle ABC$? Odgovor utemelji!



$$\vec{a} = \vec{AB} = \begin{bmatrix}-3\\3\\0\end{bmatrix}$$

$$\vec{b} = \vec{AC} = \begin{bmatrix}-5\\0\\-5\end{bmatrix}$$

$$\frac{1}{3} + \frac{2}{5} = \frac{5+6}{15} = \frac{11}{15} < 1$$

$$\vec{AD} = \begin{bmatrix}-3\\1\\-2\end{bmatrix} = \alpha\vec{a} + \beta\vec{b} = \begin{bmatrix}-3\alpha-5\beta\\3\alpha\\-5\beta\end{bmatrix} \begin{matrix}\checkmark\\\checkmark\\\checkmark\end{matrix}$$

$3\alpha = 1$, $\alpha = \frac{1}{3}$ $\quad -5\beta=-2$, $\beta = \frac{2}{5}$ $\quad -3\alpha-5\beta = -1-2=-3$ ✓

$\vec{AD} = \frac{1}{3}\vec{a} + \frac{2}{5}\vec{b} \Rightarrow$ D leži znotraj $\triangle ABC$, ker je $0 \le \alpha \le 1$ in $\alpha+\beta < 1$.

2. naloga (35 točk)

Dani sta matriki A in B ter vektorja $\vec{c}$ in $\vec{d}$:

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

a) (15) Ali sta sistema $A\vec{x} = \vec{c}$ in $A\vec{x} = \vec{d}$ rešljiva? Zakaj oziroma zakaj ne? Za vsakega od sistemov, ki je rešljiv, poišči vse rešitve.

$$\left[\begin{array}{cccc|cc} 1 & -1 & 2 & 3 & 5 & 3 \\ -2 & 2 & 1 & -1 & 4 & 4 \\ -1 & 1 & 2 & 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cccc|cc} 1 & -1 & 2 & 3 & 5 & 3 \\ 0 & 0 & 5 & 5 & 14 & 10 \\ 0 & 0 & 4 & 4 & 8 & 8 \end{array}\right] \sim \left[\begin{array}{cccc|cc} 1 & -1 & 2 & 3 & 5 & 3 \\ 0 & 0 & 1 & 1 & 14/5 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{array}\right]$$

$A\vec{x} = \vec{c}$ ni rešljiv

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 1 & -1 \\ 0 & 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x_1 = x_2 - x_4 - 1 \\ \to x_3 = -x_4 + 2 \end{array}$$

$$\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

vse rešitve $A\vec{x} = \vec{d}$

b) (5) Poišči bazo in določi dimenzijo $C(A)$, stolpčnega prostora matrike A.

$$A \sim \begin{bmatrix} \boxed{1} & -1 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

baza $C(A)$ sta $\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ in $\vec{a}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $\quad \dim C(A) = 2$

c) (5) Poišči bazo in določi dimenzijo $N(A)$, ničelnega prostora matrike A.

$$A \sim \begin{bmatrix} \boxed{1} & -1 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \to x_1 = x_2 - x_4 \\ \to x_3 = -x_4 \end{array} \qquad \begin{bmatrix} x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}$$

$$\vec{n}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{n}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

ista kot pri a) !

baza: $\{\vec{n}_1, \vec{n}_2\}$ $\quad \dim N(A) = 2$

d) (10) Poišči matriko X, ki reši matrično enačbo $BX = A$.

$B^{-1} \cdot \backslash \; BX = A$

$X = B^{-1}A$ (če B^{-1} obstaja)

$$[B \mid I] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 2 \\ 0 & 2 & 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -1 & -2 & 2 \\ 1 & 1 & -1 \\ -2 & -1 & 2 \end{matrix}}_{B^{-1} \text{ obstaja}} \\ 0 & 0 & 1 & \end{array}\right] \sim$$

$$X = B^{-1}A = \begin{bmatrix} -1 & -2 & 2 \\ 1 & 1 & -1 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ -2 & 2 & -1 & -3 \end{bmatrix}$$

3. naloga (30 točk)

Preslikava $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^4$ je podana s predpisom $\varphi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_3 \\ 0 \\ -3x_1 + 3x_2 \\ x_2 + 3x_3 \end{bmatrix}$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

a) (8) Ali je φ linearna preslikava?

$$\underline{\underline{\varphi(\vec{x}+\vec{y})}} = \varphi\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) = \begin{bmatrix} x_1+y_1+3(x_3+y_3) \\ 0 \\ -3(x_1+y_1)+3(x_2+y_2) \\ x_2+y_2+3(x_3+y_3) \end{bmatrix} = \begin{bmatrix} x_1+3x_3+y_1+3y_3 \\ 0 \\ -3x_1+3x_2-3y_1+3y_2 \\ x_2+3x_3+y_2+3y_3 \end{bmatrix} =$$

$$= \begin{bmatrix} x_1+3x_3 \\ 0 \\ -3x_1+3x_2 \\ x_2+3x_3 \end{bmatrix} + \begin{bmatrix} y_1+3y_3 \\ 0 \\ -3y_1+3y_2 \\ y_2+3y_3 \end{bmatrix} = \underline{\underline{\varphi(\vec{x})+\varphi(\vec{y})}} \text{ za vse } \vec{x},\vec{y} \longrightarrow \text{je aditivna}$$

$$\underline{\underline{\varphi(\alpha\vec{x})}} = \varphi\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1+3\alpha x_3 \\ 0 \\ -3\alpha x_1+3\alpha x_2 \\ \alpha x_2+3\alpha x_3 \end{bmatrix} = \begin{bmatrix} \alpha(x_1+3x_3) \\ \alpha\cdot 0 \\ \alpha(-3x_1+3x_2) \\ \alpha(x_2+3x_3) \end{bmatrix} = \alpha\begin{bmatrix} x_1+3x_3 \\ 0 \\ -3x_1+3x_2 \\ x_2+3x_3 \end{bmatrix} = \underline{\underline{\alpha\varphi(\vec{x})}} \text{ za vse } \vec{x},\alpha$$

$\downarrow$

je homogena

$\Rightarrow$ je linearna

b) (8) Poišči matriko M, ki pripada φ glede na standardni bazi $\mathbb{R}^3$ in $\mathbb{R}^4$.

$$\varphi(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix} \quad \varphi(\vec{e}_2) = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} \quad \varphi(\vec{e}_3) = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix} \longrightarrow \underline{\underline{M = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}}}$$

c) (7) Poišči bazo za $\ker\varphi$.

$$M = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & 0 & 0 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \longrightarrow x_1 = -3x_3 \\ \\ \longrightarrow x_2 = -3x_3 \\ \\ \end{matrix}$$

lin. neod. stolpca

$$\begin{bmatrix} -3x_3 \\ -3x_3 \\ x_3 \end{bmatrix} \qquad \vec{m}_1 = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix} \qquad \text{baza: } \underline{\underline{\{\vec{m}_1\}}}$$

d) (7) Kateri stolpci matrike M tvorijo bazo za $\operatorname{im}\varphi$? Zapiši jo.

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix} \text{ in } \vec{a}_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \text{ baza je } \{\vec{a}_1, \vec{a}_2\}$$

$$\vec{a}_3 = 3(\vec{a}_1 + \vec{a}_2)$$