Network growth by copying

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We introduce a growing network model in which a new node attaches to a randomly selected node, as well as to all ancestors of the target node. This mechanism produces a sparse, ultrasmall network where the average node degree grows logarithmically with network size while the network diameter equals 2. We determine basic geometrical network properties, such as the size dependence of the number of links and the in- and out-degree distributions. We also compare our predictions with real networks where the node degree also grows slowly with time—the Internet and the citation network of all *Physical Review* papers.

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I. INTRODUCTION

Many networks in nature and technology are sparse, i.e., the average node degree is much smaller than the total number of nodes N [1,2]. Widely studied classes of networks, such as regular grids, random graphs, and scale-free networks, are maximally sparse, as the average node degree remains finite as $N\to\infty$. However, in other examples of real sparse networks, such as the Internet, the average node degree grows, albeit very slowly, with system size. Motivated by this observation, we introduce a simple network growth mechanism of copying that naturally generates sparse networks in which the average node degree diverges logarithmically with system size. We dub these log-networks. Analogous results appear in previous investigations of related models [3,4], while models where a slowly increasing ratio of links to nodes is imposed externally have also been considered [5,6].

To motivate the copying mechanism for log-networks, let us recall the growing network with redirection (GNR) [7]. The GNR is built by adding nodes according to the following simple rule. Each new node initially selects an earlier "target" node at random. With a specified probability, a link from the new node to the target node is created; with a complementary probability, the link is redirected to the ancestor node of the target. Although the target node is chosen randomly, the redirection mechanism generates an effective preferential attachment because a high-degree node is more likely to be the ancestor of a randomly selected node. By this feature, redirection leads to a power-law degree distribution for the network. The GNR thus provides an appealingly simple mechanism for preferential attachment, as well as an extremely efficient way to simulate large scale-free networks [8].

The growing network with redirection is a simplification of a previous model [9] which was proposed to mimic the copying of links in the world-wide web. In this web model, a new node links to a randomly chosen target node and also to its ancestor nodes (subject to a constraint on the maximum number of links created). In the context of citations, copying is (regrettably) even more natural, as it is easier merely to copy the references of a cited paper rather than to look at the original references [10]. As the literature grows, the copying mechanism will necessarily lead to later publications having more references than earlier publications.

In the following sections, we analyze a growing network model with copying (GNC). We consider a model with no global bound on the number of links emanating from a new node. We shall see that this simple copying mechanism generates log-networks. We will use the master equation approach to derive basic geometric properties of the network. We then compare our prediction about the logarithmic growth of the average degree with data from *Physical Review* citations.

II. GNC MODEL

We now define the GNC model precisely. The network grows by adding nodes one at a time. A newly introduced node randomly selects a target node and links to it, as well as to all ancestor nodes of the target node (Fig. 1).

If the target node is the initial root node, no additional links are generated by the copying mechanism. If the newly

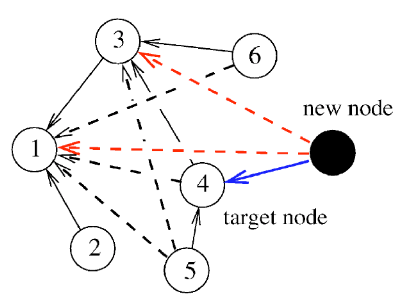

FIG. 1. Illustration of the growing network with copying (GNC). The time order of the nodes is indicated. Initial links are solid and links to ancestor nodes are dashed. Later links partially obscure earlier links. The new node initially attaches to random target node 4, as well as to its ancestors, 1 and 3.

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introduced node were to always choose the root node as the target, a star graph would be generated. On the other hand, if the target node is always the most recent one in the network, all previous nodes are ancestors of the target and the copying mechanism would give a complete graph. Correspondingly, the total number of links L_N in a network of N nodes can range from $N-1$ (star graph) to $N(N-1)/2$ (complete graph). Notice also that the number of outgoing links from each new node (the out-degree) can range between 1 and the current number of nodes.

III. NETWORK STRUCTURE

We now study geometric properties of the GNC model by the master equation approach. We determine how the total number of links L grows with N, as well as the in-degree, out-degree, and the joint in/out-degree distributions.

A. Total number of links

Let $L(N)$ be the average value of the total number of links in a network of N nodes. If a newly introduced node selects a target node with j ancestors, then the number of links added to the network will be $1+j$. Therefore, the average total number of links satisfies

$$L(N+1) = L(N) + \frac{1}{N}\left\langle \sum_{\alpha} (1+j_\alpha) \right\rangle = L(N) + 1 + \frac{L(N)}{N}. \tag{1}$$

The factor N^{-1} in the first line assures that a target node α is selected uniformly from among all N nodes, and we obtain the second line by employing the sum rule $\langle \Sigma_\alpha j_\alpha \rangle = L$.

Dividing Eq. (1) by $N+1$ gives

$$\frac{L(N+1)}{N+1} - \frac{L(N)}{N} = \frac{1}{N+1},$$

and then summing both sides from 1 to $N-1$ gives the solution

$$L(N) = N(H_N - 1). \tag{2}$$

Here $H_N = \Sigma_{n=1}^{N} n^{-1}$ is the harmonic number. [For concreteness, we assume that the network starts with a single node, so that $L(1)=0$.] Using the asymptotics of the harmonic numbers [11], we find

$$L(N) = N \ln N - N(1-\gamma) + \frac{1}{2} - \frac{1}{12N} + \cdots , \tag{3}$$

where $\gamma = 0.577\,215\,66\ldots$ is the Euler constant. The leading asymptotic behavior of $N \ln N$ can also be obtained more easily from Eq. (1) by taking the continuum approximation and solving the resulting differential equation.

Thus we conclude that the average degree of the network grows logarithmically with the system size; that is, the copying mechanism generates a log-network. This simple phenomenon is one of our major results.

We now briefly discuss the probability distribution of the total number of links $P_N(L)$ for a network of N nodes. Simulations show that the distribution is asymmetric and quite broad (Fig. 2). To understand the origin of the asymmetry, notice that both the extreme cases of the star graph $(L=N-1)$ and the complete graph $[L=N(N-1)/2]$ each occur with probability

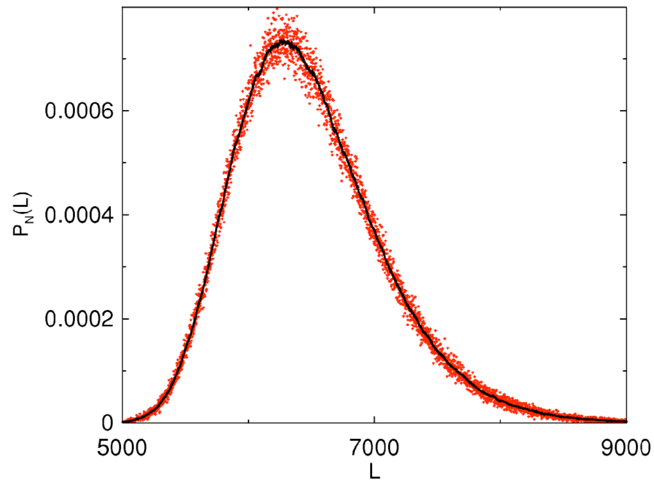

FIG. 2. Distribution of the number of links $P_N(L)$ for 10^5 realizations of a GNC of $N=1000$ sites. Shown are both the raw data and the result of averaging the data over a 1% range. For this value of N, the mean number of links is 6485.56, while Eq. (3) gives 6485.47… .

$$P_N(L) = \frac{1}{(N-1)!},$$

because each new node must select one specific target node. Therefore, the distribution of the total number of links vanishes much more sharply near the lower cutoff.

Near the peak, however, the distribution $P_N(L)$ is symmetric about the average. More precisely, when the deviation from the average $L(N) = \Sigma_L L P_N(L)$ is of the order of $\Sigma(N) = \sqrt{\Sigma_L [L - L(N)]^2 P_N(L)}$, the distribution approaches a symmetric Gaussian shape. The value of the standard deviation as $N \to \infty$ is

$$\Sigma(N) \to CN, \quad C = \sqrt{2 - \pi^2/6} = 0.595\,874\ldots , \tag{4}$$

as derived in Appendix A. The relative width of the distribution is measured by the standard deviation $\Sigma(N)$ divided by the average $L(N)$; this ratio approaches zero as $(\ln N)^{-1}$ when $N \to \infty$, so that fluctuations die out slowly. This slow decay of fluctuations explains why the distribution (Fig. 2) remains wide for large N and why it looks asymmetric near the peak.

The GNC model can be extended to allow for wider copying variability. For example, instead of linking to one initial random target node, we can link to m random initial targets. Further, we can link to each target node with probability p and to each of the corresponding ancestors with probability q. For this general (m,p,q) model, the analog of Eq. (1) is [12]

$$L(N+1) = L(N) + \frac{m}{N}\left\langle \sum_{\alpha} (p + qj_{\alpha}) \right\rangle \qquad (5)$$

which reduces to $dL/dN = mp + mq(L/N)$ in the continuum approximation. The asymptotic growth of the average total number of links crucially depends on the parameter mq,

$$L(N) = \begin{cases} \frac{mp}{1-mq}N & \text{for } mq < 1, \\ mpN \ln N & \text{for } mq = 1, \\ \propto N^{mq} & \text{for } mq > 1. \end{cases} \qquad (6)$$

Thus incomplete copying leads to an average node degree that is independent of N when $mq < 1$, while marginal logarithmic dependence is recovered when $mq = 1$. There is also a pathology for $mq > 2$, as the number of links in the network would exceed that of a complete graph with the same number of nodes. In this case, it is not possible to accommodate all the links specified by the copying rule without having a multigraph, i.e., allowing for more than one link between a given pair of nodes.

B. Comparison with empirical data

We now present empirical data from the citation network of *Physical Review* to test whether the average degree of these networks grows with time, and if so, whether the growth is consistent with log-networks. Data from all issues of *Physical Review* journals are available, encompassing a time span of 110 years [13]. From these data, we have the following evidence that citations may be described as a log-network. Specifically, the average number of references in the reference list of each *Physical Review* paper grows systematically with time and is consistent with a linear increase (Fig. 3). Additionally, the number of *Physical Review* papers published in a given year roughly grows exponentially with time [13]. Thus the cumulative number of *Physical Review* papers up to a given year also grows exponentially. As a result, the number of references should grow logarithmically with the total number of available papers. This behavior is reasonably consistent with the data of Fig. 3.

In a related vein, the average number of coauthors per paper has grown slowly with time, due in part to the growing trend for collaborative research and the continuing ease of long-distance scientific interaction. While coauthorship and other collaboration networks have recently been investigated (see, e.g., [14–16]), the analysis has primarily been on network properties at a fixed time. There is, however, one study of the number of mathematics papers with one, two, and more authors since 1940 [17]. These data show that the fraction of singly authored papers is decreasing systematically, while the number of multiple-authored papers is steadily growing. Thus it should be interesting to track the time dependence of the number of coauthors in scientific publications from the current studies of collaboration networks.

Interestingly, the Internet and the world wide web exhibit certain similarities with log-networks. For example, the total number of links exceeds the total number of nodes in the world wide web by about an order of magnitude [18,19].

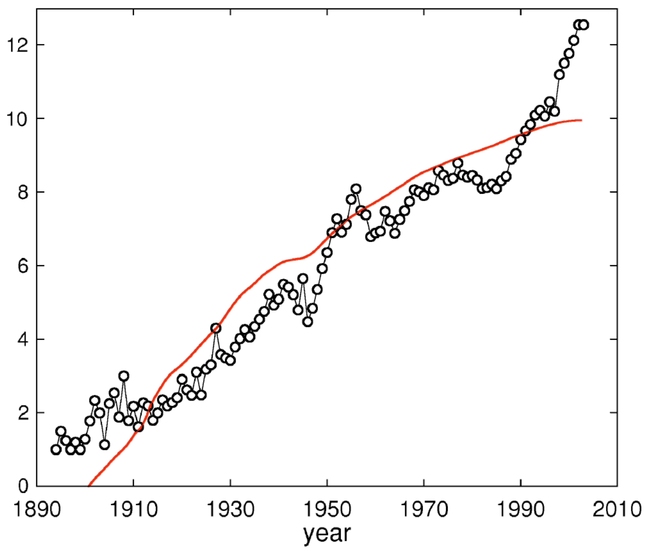

FIG. 3. (Color online) Average number of references in the reference list of *Physical Review* papers published in each year (○). Also shown as a smooth curve is the logarithm of the cumulative number of *Physical Review* papers that were published up to each year. The value 5 is subtracted from the latter data to make the two datasets lie in the same range.

Similarly for the Internet, specifically for the Autonomous Systems (AS) graph, the average number of links per node is also growing slowly but systematically with time [20]. Qualitatively these behaviors are consistent with our expectations from log-networks. It is still not possible to reach definitive conclusions about the precise growth rate on N since the available data for the AS graph [20] cover a time period when the total number of ASs has increased only by a factor of 4 (from $N = 3060$ in 1997 to $N = 12\,155$ in 2001).

C. In-degree distribution

By its very construction, the links of the GNC network are directed and there is an in-degree i and an out-degree j for each node (Fig. 4), and thus two distinct corresponding degree distributions. In this subsection, we study the in-degree distribution.

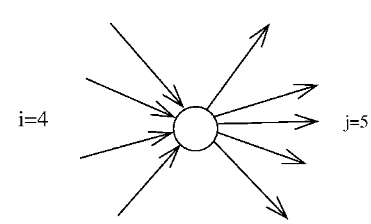

FIG. 4. A node with in-degree (number of incoming links) $i = 4$, out-degree (number of outgoing links) $j = 5$, and total degree $k = 9$.

Let $P_i(N)$ be the average number of nodes with in-degree i in a network consisting of N total nodes. This distribution satisfies

$$P_i(N+1) = P_i(N) - \frac{i+1}{N}P_i(N) + \frac{i}{N}P_{i-1}(N) + \delta_{i,0}. \quad (7)$$

The loss term accounts for the following two processes: (a) either a node of in-degree i, or (b) any of its i daughter nodes was chosen as the target. Either of these processes leads to the loss of a node with in-degree i. The total loss rate of $P_i(N)$ is thus $(i+1)/N$. The gain term is explained similarly, and the last term on the right-hand side of Eq. (7) describes the effect of the introduction of a new node with no incoming links. Finally, notice that Eqs. (7) hold for $i \leqslant N$. When $i=N$, there is no longer a loss term and the master equation reduces to $P_N(N+1)=P_{N-1}(N)=1$. This accounts for the fact that the root node is necessarily linked to all other nodes and therefore there is one node with degree $N-1$ in a network of N nodes.

We compute the in-degree distribution by induction. Solving for the first few $P_i(N)$ for small i directly, we find a simple form for the general case that we then check solves the master equation (7). We thus find

$$P_i(N) = \frac{N}{(i+1)(i+2)} \text{ for } i < N-1, \quad (8)$$

while $P_i(N)=0$ for $i \geqslant N$.

The asymptotic i^{-2} decay agrees with the logarithmic divergence for the average node degree from the previous section, and perhaps explains the proliferation of exponent values close to 2 for the in-degree distribution that are observed in empirical studies of collaboration networks [14–16] and in the world wide web [18,19,21–23]. In particular, a comprehensive study by Broder *et al.* [18] reports an exponent value 2.09, while a recent work by Donato *et al.* [19] (relying on the WebBase project at Stanford [24]) quotes an exponent value 2.1.

D. Out-degree distribution

To determine the out-degree distribution, it is helpful to think of the network as a genealogical tree, as illustrated in Fig. 5 (see also [7] for this construction). Initially the network consists of one root node. Subsequent nodes that attach to the root node will have out-degree 1 and lie in the first layer. Similarly, a new node that attaches to a node with out-degree 1 lies in the second layer. By the copying mechanism, nodes in this second layer also link to the root and therefore have out-degree 2. Nodes in the nth layer directly attach to a node in the $(n-1)$st layer and, by virtue of copying, also attach to one node in every previous layer. Thus nth layer nodes have out-degree n. We now use this genealogical tree picture to determine the out-degree distribution of the network.

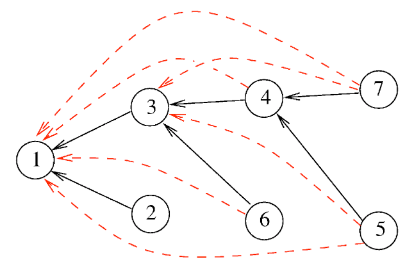

FIG. 5. Genealogical tree representation of the network of Fig. 1, with nodes arranged in layers left to right according to their out-degree. The initial layer contains only the root node. The number of nodes in subsequent layers increases as the network grows. The initial links are shown as solid arrows and the copied links as dashed arrows.

Let $Q_j(N)$ be the average number of nodes with out-degree j in a network consisting of N nodes. By definition, $Q_0(N) \equiv 1$. On the other hand, the number of nodes with out-degree $j \geqslant 1$ grows each time a node with out-degree $j-1$ is selected as the target node. The out-degree distribution thus satisfies the master equation

$$Q_j(N+1) = Q_j(N) + \frac{1}{N}Q_{j-1}(N). \quad (9)$$

This equation applies even for $j=0$ if we set $Q_{-1}(N) \equiv 0$. Using the recursive nature of these equations, we first solve for $Q_1(N)$, then $Q_2(N)$, etc., and ultimately the out-degree distribution for all j. This procedure gives

$$Q_j(N+1) = \sum_{1 \leqslant m_1 < \cdots < m_j \leqslant N} \frac{1}{m_1 \times \cdots \times m_j}. \quad (10)$$

Equivalently, we can recast the j-fold sums into simple sums, although the results look less neat. For example,

$$Q_2(N+1) = \frac{1}{2}[(H_N)^2 - H_N^{(2)}],$$

where $H_N^{(2)} = \sum_{n=1}^{N} n^{-2}$. The asymptotic behaviors of H_N, $H_N^{(2)}$, and other generalized harmonic numbers are known [11], and the resulting asymptotics of the out-degree distribution are

$$Q_1(N+1) = H_N = \ln N + \gamma + \frac{1}{2N} - \frac{1}{12N^2} + \cdots ,$$

$$Q_2(N+1) = \frac{1}{2}(\ln N)^2 + \gamma \ln N + \frac{1}{2}\left[\gamma^2 - \frac{\pi^2}{6}\right] + \cdots$$

and analogous results hold for $Q_j(N)$ for larger j.

If we merely want to establish the leading asymptotic behavior, we can replace the summation in Eq. (10) by integration. This then leads to the simple result

$$Q_j(N) \to \frac{(\ln N)^j}{j!}. \quad (11)$$

Alternatively, we can derive this result within a continuum approach by replacing finite differences by derivatives in the large-N limit of Eq. (9). The procedure recasts the discrete master equations into the differential equations

$$\frac{dQ_j}{dN} = \frac{1}{N} Q_{j-1}(N)$$

whose solution is indeed Eq. (11).

The Poisson form of the out-degree distribution contradicts the commonly presumed power-law form. There is previous literature by Broder *et al.* that suggested that the out-degree distribution has a power-law tail, with an exponent close to 2.7 [18]. However, this work also noted that a power law is not a good fit to the data and that the out-degree distribution may possibly follow a Poisson distribution. In fact, the analysis of Ref. [19] that is based on more recent data on the structure of the web [24] convincingly shows that a power law does not fit the out-degree distribution.

E. Joint degree distribution

We define the joint degree distribution $N_{i,j}(N)$ as the average number of nodes with in-degree i and out-degree j in a network of N nodes. The in- and out-degree distributions can then be distilled from the joint distribution via $P_i(N) = \Sigma_j N_{i,j}(N)$ and $Q_j(N) = \Sigma_i N_{i,j}(N)$. Furthermore, the average number of nodes with total degree k is simply given by $N_k(N) = \Sigma_{i+j=k} N_{i,j}(N)$.

The joint degree distribution satisfies

$$N_{i,j}(N+1) = N_{i,j}(N) + \frac{i}{N} N_{i-1,j}(N) - \frac{i+1}{N} N_{i,j}(N) + \frac{1}{N} Q_{j-1}(N)\,\delta_{i,0}, \tag{12}$$

which is an obvious generalization of the governing equations (7) and (9) for the separate in- and out-degree distributions. Because of the presence of the last out-degree term on the right-hand side of Eq. (12), the scaling of the joint degree distribution with system size does not hold: $N_{i,j}(N) \neq N n_{i,j}$. Therefore, we cannot reduce Eq. (12) to an N-independent recursion.

Nevertheless, Eqs. (12) still have the important simplifying feature of being recursive and thus soluble in an inductive fashion. Thus, for example, for $i=0$ we have

$$N N_{0,j}(N+1) = (N-1) N_{0,j}(N) + Q_{j-1}(N)$$

from which

$$N_{0,j}(N+1) = \frac{1}{N} \sum_{M=1}^{N} Q_{j-1}(M). \tag{13}$$

For $i \geq 1$, we rewrite Eqs. (12) in the form

$$N N_{i,j}(N+1) = (N-i-1) N_{i,j}(N) + i N_{i-1,j}(N). \tag{14}$$

Now the substitution

$$N_{i,j}(N) = \frac{\Gamma(N-i-1)\Gamma(i+1)}{\Gamma(N)} A_{i,j}(N) \tag{15}$$

reduces Eq. (14) to the constant-coefficient recursion

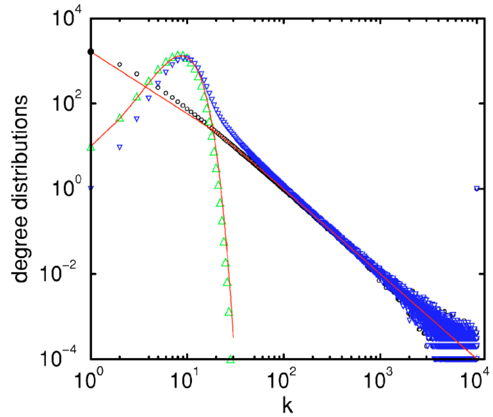

FIG. 6. The in- (○), out- (△), and total (▽) degree distributions for 10^4 realizations of a network of $N=10^4$ nodes. Notice that there is always one node with total degree equal to 1 and one node with in- and total degree equal to N. The smooth curve that follows in in-degree data is the asymptotic prediction (8), while the curve that follows the out-degree data is the asymptotic prediction (11).

$$A_{i,j}(N+1) = A_{i,j}(N) + A_{i-1,j}(N)$$

that allows us to express $A_{i,j}$ via $A_{i-1,j}$,

$$A_{i,j}(N+1) = \sum_{M=1}^{N} A_{i-1,j}(M). \tag{16}$$

From Eq. (15), we find $A_{0,j}(N+1) = N^{-1} N_{0,j}(N+1)$ which, in conjunction with Eq. (13), gives

$$A_{0,j}(N+1) = \sum_{M=1}^{N} Q_{j-1}(M). \tag{17}$$

Therefore, starting with $A_{0,j}$ from Eq. (17), we find all the $A_{i,j}$ via Eq. (16). The final result is

$$A_{i,j}(N) = \sum_{1 \leq m_0 < \cdots < m_i < N} Q_{j-1}(m_0). \tag{18}$$

Equations (10), (15), and (18) give the full solution for the joint degree distribution.

While the complete solution is cumbersome, it can be simplified as $N \to \infty$. In this limit, we can first replace the factor $\Gamma(N-i-1)/\Gamma(N)$ by N^{-i-1} in Eq. (15). Additionally, we can replace the summation in Eq. (18) by integration. These two replacements are justified when $i \ll \sqrt{N}$. Finally, using Eq. (11) and after some algebra, we find the leading behavior

$$N_{0,j}(N) \to \frac{(\ln N)^{j-1}}{(j-1)!}$$

and more generally

$$N_{i,j}(N) \to \left[\frac{(\ln N)^{j-1}}{(j-1)!}\right]^{i+1}. \tag{19}$$

Because the Poisson form the out-degree distribution holds only when $j \ll \ln N$, the generalized Poisson form (11) for the joint degree distribution is also valid only for $j \ll \ln N$.

Finally, although the total degree distribution $N_k(N)$ does not satisfy a closed equation, we can obtain this distribution indirectly. When k is of the order of $\ln N$ or smaller, we can use Eq. (19) to find $N_k(N)$. The situation in the range $k \gg \ln N$ is even simpler: In this region, the total degree distribution essentially coincides with the in-degree distribution and therefore $N_k(N) \to Nk^{-2}$ (Fig. 6).

IV. SUMMARY

We introduced a growing network model that is based on node addition plus a simple copying mechanism—the GNC—that leads to an average node degree growing logarithmically with the total number of nodes N. This feature may account for the intriguing phenomenon observed in many real networks that the number of links increases slightly faster than the number of nodes. Copying arises naturally in the context of citations; a not untypical scenario is that an author will be familiar with a few primary references, but may simply copy secondary references from primary ones.

We solved the underlying master equations for the GNC model and showed that the in-degree distribution is a power-law over its entire range, while the out-degree distribution is asymptotically Poissonian. The total degree distribution is consequently a hybrid of the power-law and Poisson forms. There is, on average, one node with total degree equal to 1, and there is always one node—the root—that has in-degree equal to $N-1$. Thus the node degree ranges from 1 to $N-1$. Since the distribution of L has a width that scales linearly with N while $L(N)$ grows as $N \ln N$, fluctuations in node degree are appreciable even for very large networks. Finally, each node is connected to the root, so that the network diameter equals 2, independent of N.

From long-term *Physical Review* publication data, the average number of references per paper (the out degree) grows slowly with the total literature size, consistent with the logarithmic growth predicted by the GNC model. However, this growth in the GNC model is not robust when parameters that quantify the extent of copying are varied. The apparent logarithmic growth for the average number of references per paper in *Physical Review* is thus a bit surprising, and it will be worthwhile to test whether logarithmic growth arises in a wider range of empirical networks.

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APPENDIX: FLUCTUATIONS

In this appendix, we find the variance in the distribution of the number of links $P_N(L)$. We start by computing the first two moments of the out-degree distribution, $u_N \equiv \langle j \rangle = \Sigma_j j Q_j(N)/N$ and $v_N \equiv \langle j^2 \rangle$. We then use these results to derive the variance of $P_N(L)$.

To determine u_N and v_N, we can in principle use Eq. (10). However, a direct approach is more useful. Starting with $Nu_N = \langle \Sigma j \rangle$, we find that adding a new node leads to the recursion relation

$$(N+1)u_{N+1} = \left\langle 1 + j_\alpha + \sum j \right\rangle = 1 + u_N + Nu_N, \qquad \text{(A1)}$$

which is nothing but Eq. (1). In a similar manner, we derive a recursion relation for $Nv_N = \langle \Sigma j^2 \rangle$,

$$(N+1)v_{N+1} = \left\langle (1+j_\alpha)^2 + \sum j^2 \right\rangle = 1 + 2u_N + v_N + Nv_N,$$

which reduces to

$$v_{N+1} = v_N + \frac{2}{N+1}u_N + \frac{1}{N+1}. \qquad \text{(A2)}$$

The variance $\sigma^2(N) = v_N - (u_N)^2$ therefore satisfies

$$\sigma^2(N+1) = \sigma^2(N) + \frac{1}{N+1} - \frac{1}{(N+1)^2}. \qquad \text{(A3)}$$

From this simple recursion, we get

$$\sigma^2(N) = H_N - H_N^{(2)}. \qquad \text{(A4)}$$

The relative magnitude of fluctuations dies out slowly, as the standard deviation $\sigma(N) \sim \sqrt{\ln N}$ divided by the average $u_N \sim \ln N$ approaches zero as $(\ln N)^{-1/2}$.

Consider now the average number of links in the network $L(N) = \langle \Sigma j \rangle = Nu_N$ and the corresponding second moment $L_2(N) \equiv \langle L_N^2 \rangle$. After the addition of a new node, the second moment changes according to

$$\begin{aligned} L_2(N+1) &= \left\langle \left(1 + j_\alpha + \sum j\right)^2 \right\rangle \\ &= \left\langle (1+j_\alpha)^2 + 2\sum j + 2j_\alpha \sum j + \left(\sum j\right)^2 \right\rangle \\ &= 1 + 2u_N + v_N + 2Nu_N + \left(1 + \frac{2}{N}\right) L_2(N). \end{aligned}$$

Now we use $L(N) = \langle \Sigma j \rangle = Nu_N$ to write the square of Eq. (A1) in the form

$$L(N+1)^2 = \left(1 + \frac{2}{N}\right) L(N)^2 + (u_N)^2 + 2(N+1)u_N + 1,$$

and then subtracting this from the previous equation, the variance $\Sigma^2(N) = L_2(N) - L(N)^2$ satisfies

$$\Sigma^2(N+1) = \left(1 + \frac{2}{N}\right)\Sigma^2(N) + \sigma^2(N). \qquad \text{(A5)}$$

The homogeneous part of Eq. (A5) suggests seeking a solution of the form $\Sigma^2(N) = N(N+1)S_N$. This substitution recasts Eq. (A5) into $S_{N+1} - S_N = [(N+1)(N+2)]^{-1}\sigma^2(N)$. This is an exact discrete first derivative for S_N. Hence S_N equals

$\Sigma_1^{N-1}[(M+1)(M+2)]^{-1}\sigma^2(M)$. Thus the variance is

$$\Sigma^2(N) = N(N+1)\sum_{M=1}^{N-1}\frac{\sigma^2(M)}{(M+1)(M+2)}. \qquad (A6)$$

Finally, by substituting $\sigma^2(M)=\Sigma_{j\leqslant M}(j^{-1}-j^{-2})$ from Eqs. (A4) into (A6), and changing the order of the two sums, we find that $\Sigma^2(N)\to\left(2-\frac{1}{6}\pi^2\right)N(N+1)$ as $N\to\infty$. This leads to the asymptotic expression for the standard deviation given in Eq. (4).

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